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Dynamics of a Rolling Coin

Let us consider the motion of a coin rolling on a horizontal table without slipping. This "rolling without slipping" constraint cannot be expressed as a function of coordinates, and is called a non-holonomic constraint, which makes the coin's motion complex.

The parameters characterizing the system are the mass M and the moment of inertia tensor $\hat{I}$ about the center of mass, the radius of the coin a, and the gravitational constant g.

1 Equations of Motion

With the center of mass velocity $\boldsymbol{v}$ and the angular velocity $\boldsymbol{\omega}$ of the coin, the equations of motion are

$$M\frac{d\boldsymbol{v}}{dt} = M\boldsymbol{g} + \boldsymbol{R} \qquad \text{Translational motion} \tag{1}$$

$$\frac{d}{dt}\left(\hat{I}\boldsymbol{\omega}\right) = \boldsymbol{c} \times \boldsymbol{R} \qquad \text{Rotational motion} \tag{2}$$

These are two vector equations. Here, $\boldsymbol{g}$ is the gravitational acceleration vector of magnitude g pointing vertically downward, $\boldsymbol{R}$ is the reaction force from the floor, and the vector $\boldsymbol{c}$ points from the center of the coin to the contact point with the floor.

The constraint of rolling without slipping relates the center of mass velocity $\boldsymbol{v}$ and the angular velocity $\boldsymbol{\omega}$ as follows:

$$\boldsymbol{v} = -\boldsymbol{\omega} \times \boldsymbol{c}. \qquad \text{(no-slip condition)} \tag{3}$$

This cannot be expressed by the relation of coordinates and angles, so it is a non-holonomic constraint.

By eliminating $\boldsymbol{R}$ from Eqs. (1) and (2), and using Eq. (3) to express $\boldsymbol{v}$ in terms of $\boldsymbol{\omega}$, the equation of motion becomes

$$\frac{d}{dt}\left(\hat{I}\boldsymbol{\omega}\right) = M\boldsymbol{c} \times \left(-\frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{c}) - \boldsymbol{g}\right). \tag{4}$$

2 Coordinate Systems

Here, we introduce two coordinate systems, xyz and XYZ.

(i) xyz: a inertial coordinate system. The horizontal floor is the z-x plane, and the vertical upward direction is taken as the y-axis.

(ii) XYZ: the origin at the center of the coin, with the Z-axis perpendicular to the coin, X-axis lies in the horizontal plane, and the Y-axis is upward in the coin plane.

The basis vectors, i.e., the unit vectors parallel to each axis, are denoted by

$$\left(\boldsymbol{e}_x,\ \boldsymbol{e}_y,\ \boldsymbol{e}_z\right), \qquad \left(\boldsymbol{e}_X,\ \boldsymbol{e}_Y,\ \boldsymbol{e}_Z\right), \tag{5}$$

where $\boldsymbol{e}_X$ and $\boldsymbol{e}_Y$ are defined as

$$\boldsymbol{e}_X \equiv \frac{\boldsymbol{e}_y \times \boldsymbol{e}_Z}{|\boldsymbol{e}_y \times \boldsymbol{e}_Z|}, \quad \boldsymbol{e}_Y \equiv \boldsymbol{e}_Z \times \boldsymbol{e}_X. \tag{6}$$

Furthermore, we define three angles (θ, ϕ, ψ) to represent the orientation of the coin. That is, when $(\theta, \phi, \psi) = 0$, the coin lies in the xy-plane, and the XYZ axes are parallel to the xyz axes. The coin in orientation (θ, ϕ, ψ) is obtained from the $(0, 0, 0)$ orientation by

1. rotating by ψ around the z-axis,
2. rotating by θ around the x-axis,
3. rotating by ϕ around the y-axis,

in that order.

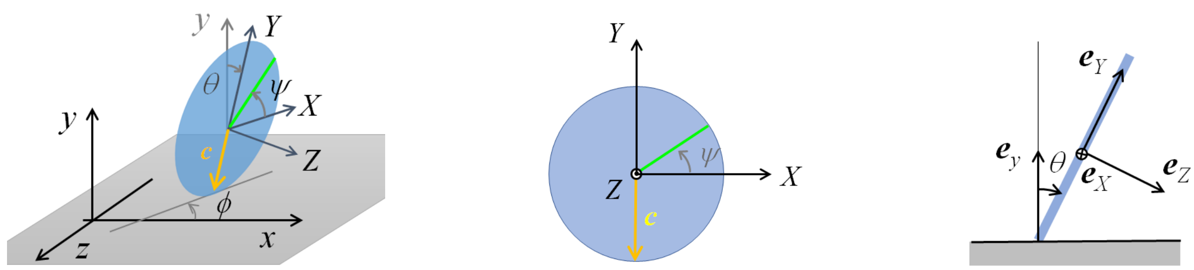

図 1: Two coordinate systems xyz and XYZ, and three angles (θ, ϕ, ψ) that represent the coin's orientation (left figure). View from the Z axis (center) and the $-X$ axis (right). When $(\theta, \phi) = 0$, the two coordinate systems coincide.

Then, the relationship between basis vectors can be expressed by the rotation matrix

$$\hat{R}(\theta,\phi) \equiv \begin{pmatrix} \cos\phi, & 0, & -\sin\phi \\ \sin\theta\sin\phi, & \cos\theta, & \sin\theta\cos\phi \\ \cos\theta\sin\phi, & -\sin\theta, & \cos\theta\cos\phi \end{pmatrix} = \hat{R}(x,\theta)\hat{R}(y,\phi) \tag{7}$$

$$\hat{R}^t(\theta,\phi) = \begin{pmatrix} \cos\phi, & \sin\theta\sin\phi, & \cos\theta\sin\phi \\ 0, & \cos\theta, & -\sin\theta \\ -\sin\phi, & \sin\theta\cos\phi, & \cos\theta\cos\phi \end{pmatrix} = \hat{R}^{-1}(\theta,\phi) \tag{8}$$

as follows:

$$\begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \\ \boldsymbol{e}_Z \end{pmatrix} = \hat{R}(\theta,\phi) \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix} = \hat{R}^t(\theta,\phi) \begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \\ \boldsymbol{e}_Z \end{pmatrix} \tag{9}$$

Here, $\hat{R}(x,\theta)$ and $\hat{R}(y,\phi)$ represent the rotation matrices about the x- and y-axes by angles θ and ϕ, respectively, namely

$$\hat{R}(x,\theta) = \begin{pmatrix} 1, & 0, & 0 \\ 0, & \cos\theta, & \sin\theta \\ 0, & -\sin\theta, & \cos\theta \end{pmatrix}, \quad \hat{R}(y,\phi) = \begin{pmatrix} \cos\phi, & 0, & -\sin\phi \\ 0, & 1, & 0 \\ \sin\phi, & 0, & \cos\phi \end{pmatrix}. \tag{10}$$

Let $\boldsymbol{\Omega}$ be the angular velocity vector of the XYZ coordinate system. Then, $\boldsymbol{\Omega}$ is given by the angular velocity vector of the coin $\boldsymbol{\omega}$ as

$$\boldsymbol{\Omega} = \dot{\phi}\boldsymbol{e}_y + \dot{\theta}\boldsymbol{e}_X, \qquad \boldsymbol{\omega} = \dot{\phi}\boldsymbol{e}_y + \dot{\theta}\boldsymbol{e}_X + \dot{\psi}\boldsymbol{e}_Z. \tag{11}$$

3 Equations of Motion in the XYZ System

We express the equations of motion in the basis of the XYZ coordinate system. For a vector $\boldsymbol{u}$, the time derivative of its components in the XYZ frame is defined as

$$\left(\frac{d\boldsymbol{u}}{dt}\right)_{XYZ} \equiv \frac{du_X}{dt}\boldsymbol{e}_X + \frac{du_Y}{dt}\boldsymbol{e}_Y + \frac{du_Z}{dt}\boldsymbol{e}_Z.$$

Since the XYZ frame rotates with angular velocity $\boldsymbol{\Omega}$, the time derivative of $\boldsymbol{u}$ is given by

$$\frac{d\boldsymbol{u}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_{XYZ} + \boldsymbol{\Omega}\times\boldsymbol{u}.$$

In the XYZ system, the inertia tensor is given by

$$\hat{I} = \begin{pmatrix} \frac{1}{4}Ma^2, & 0, & 0 \\ 0, & \frac{1}{4}Ma^2, & 0 \\ 0, & 0, & \frac{1}{2}Ma^2 \end{pmatrix}_{XYZ}. \tag{12}$$

The contact vector $\boldsymbol{c}$ and the gravitational acceleration vector $\boldsymbol{g}$ are

$$\boldsymbol{c} = -a\boldsymbol{e}_Y, \qquad \boldsymbol{g} = -g\boldsymbol{e}_y.$$

Therefore, the equation of motion (4) becomes

$$\hat{I}\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ} + \boldsymbol{\Omega}\times\left(\hat{I}\boldsymbol{\omega}\right) = M\boldsymbol{c}\times\left(-\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ}\times\boldsymbol{c} + \boldsymbol{\Omega}\times\left(-\boldsymbol{\omega}\times\boldsymbol{c}\right) + g\boldsymbol{e}_y\right).$$

Rearranging this, we obtain

$$\hat{I}\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ} + Ma^2\left(\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ} - \frac{d\omega_Y}{dt}\boldsymbol{e}_Y\right) = -\boldsymbol{\Omega}\times\left(\hat{I}\boldsymbol{\omega}\right) - Ma^2\left(\Omega_Y\boldsymbol{\omega}\times\boldsymbol{e}_Y + \frac{g}{a}\boldsymbol{e}_Y\times\boldsymbol{e}_y\right).$$

The angular velocity $\boldsymbol{\omega}$ is related with the angular velocity of the XYZ coordinate system $\boldsymbol{\Omega}$ as

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\psi}\boldsymbol{e}_Z\,; \qquad \text{i.e.} \quad \omega_X = \Omega_X, \quad \omega_Y = \Omega_Y, \quad \omega_Z = \Omega_Z + \dot{\psi}. \tag{13}$$

Expanding the components of the equation of motion in the XYZ coordinate system, we obtain

$$\begin{aligned}
\left(I_X + Ma^2\right)\frac{d\omega_X}{dt} &= -\left(I_Z + Ma^2\right)\omega_Y\omega_Z - I_X\omega_Y^2\tan\theta + Ma^2\frac{g}{a}\sin\theta && (14)\\
I_X\frac{d\omega_Y}{dt} &= +I_X\omega_X\omega_Y\tan\theta + I_Z\omega_Z\omega_X && (15)\\
\left(I_Z + Ma^2\right)\frac{d\omega_Z}{dt} &= +Ma^2\omega_X\omega_Y && (16)
\end{aligned}$$

Here, $I_X = \frac{1}{4}Ma^2$, $I_Z = \frac{1}{2}Ma^2$. From Eq. (11), the angular velocity $\boldsymbol{\omega}$ can be expressed by the orientation angles (θ, ϕ, ψ) as

$$\omega_X = \dot{\theta}, \qquad \omega_Y = \dot{\phi}\cos\theta, \qquad \omega_Z = \dot{\psi} - \dot{\phi}\sin\theta,$$

from which we obtain

$$\begin{aligned}
\frac{d\theta}{dt} &= \omega_X, && (17)\\
\frac{d\phi}{dt} &= \frac{1}{\cos\theta}\,\omega_Y, && (18)\\
\frac{d\psi}{dt} &= \omega_Z + \tan\theta\,\omega_Y. && (19)
\end{aligned}$$

4 Physical Quantities

Center of Mass Velocity and Position:

$$\begin{aligned} \boldsymbol{v} &= -\boldsymbol{\omega} \times \big(-a\boldsymbol{e}_Y\big) = a\big(-\omega_Z \boldsymbol{e}_X + \omega_X \boldsymbol{e}_Z\big) \\ \dot{x} &= v_x = a\big(-\omega_Z \cos\phi + \omega_X \cos\theta \sin\phi\big) \\ \dot{y} &= v_y = -a\omega_X \sin\theta \\ \dot{z} &= v_z = a\big(\omega_Z \sin\phi + \omega_X \cos\theta\cos\phi\big) \end{aligned}$$

Contact Point with Floor:

$$\boldsymbol{r}_c = \boldsymbol{r} + \boldsymbol{c}\,; \qquad x_c = x - a\sin\theta\sin\phi, \quad z_c = z - a\sin\theta\cos\phi$$

$$\left\{ \begin{aligned} \dot{x}_c &= -a\big(\tan\theta\omega_Y + \omega_Z\big)\cos\phi = -a\dot{\psi}\cos\phi \\ \dot{z}_c &= +a\big(\tan\theta\omega_Y + \omega_Z\big)\sin\phi = +a\dot{\psi}\sin\phi \end{aligned} \right. \quad \Rightarrow \quad v_c = a|\dot{\psi}|$$

Energy:

$$\begin{aligned} E &= \frac{1}{2}Mv^2 + \frac{1}{2}\boldsymbol{\omega}^t\hat{I}\boldsymbol{\omega} + Mg(-\boldsymbol{c})\cdot\boldsymbol{e}_y \;= \frac{1}{2}Mv^2 + \frac{1}{2}\boldsymbol{\omega}^t\hat{I}\boldsymbol{\omega} + Mga\cos\theta \\ &= \frac{1}{2}Ma^2\big(\omega_X^2 + \omega_Z^2\big) + \frac{1}{2}\Big(I_X(\omega_X^2 + \omega_Y^2) + I_Z\omega_Z^2\Big) + Mga\cos\theta \end{aligned}$$

Reaction Force:

$$\begin{aligned} \boldsymbol{R} &= M\frac{d\boldsymbol{v}}{dt} + Mg\boldsymbol{e}_y \;= Ma\frac{d}{dt}\big(\boldsymbol{\omega}\times\boldsymbol{e}_Y\big) + Mg\boldsymbol{e}_y \\ &= Ma\Big(-\big(\dot{\omega}_Z - \omega_X\omega_Y\big)\boldsymbol{e}_X - \big(\omega_X^2 - \omega_Y\omega_Z\tan\theta\big)\boldsymbol{e}_Y + \big(\dot{\omega}_X + \omega_Y\omega_Z\big)\boldsymbol{e}_Z\Big) + Mg\boldsymbol{e}_y \end{aligned}$$

$$\begin{aligned} R_x &= Ma\Big(-\big(\dot{\omega}_Z - \omega_X\omega_Y\big)\cos\phi - \big(\omega_X^2 - \omega_Y\omega_Z\tan\theta\big)\sin\theta\sin\phi + \big(\dot{\omega}_X + \omega_Y\omega_Z\big)\cos\theta\sin\phi\Big) \\ R_y &= Ma\Big(-\big(\omega_X^2 - \omega_Y\omega_Z\tan\theta\big)\cos\theta - \big(\dot{\omega}_X + \omega_Y\omega_Z\big)\sin\theta\Big) + Mg \\ R_z &= Ma\Big(\big(\dot{\omega}_Z - \omega_X\omega_Y\big)\sin\phi - \big(\omega_X^2 - \omega_Y\omega_Z\tan\theta\big)\sin\theta\cos\phi + \big(\dot{\omega}_X + \omega_Y\omega_Z\big)\cos\theta\cos\phi\Big) \end{aligned}$$

$$\begin{aligned} F &\equiv \sqrt{R_x^2 + R_y^2} \;= Ma\sqrt{\big(\dot{\omega}_Z - \omega_X\omega_Y\big)^2 + \Big(\omega_X^2\sin\theta - \dot{\omega}_Z\cos\theta - \frac{\omega_Y\omega_Z}{\cos\theta}\Big)^2} \\ N &\equiv R_y \;= -Ma\Big(\omega_X^2\cos\theta + \dot{\omega}_X\sin\theta\Big) + Mg \end{aligned}$$

Coin Orientation: When $(\theta, \phi, \psi) = 0$, the coin lies in the x-z plane and the Z-axis points opposite direction to the y-axis. To orient the coin to (θ, ϕ, ψ) from this state, the following rotations should be performed in this sequence:

1. rotating by ψ around the z-axis,
2. rotating by θ around the x-axis,
3. rotating by ϕ around the y-axis,

The rotation operator $\hat{R}$ of quaternion for a rotation of angle θ about axis $\boldsymbol{n}$ is

$$\hat{R}(\boldsymbol{n}, \theta) = \cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2}; \quad \hat{n} \equiv n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k}$$

Therefore, $\hat{R}$ is given by

$$\begin{aligned} \hat{R} =& \hat{R}(\boldsymbol{e}_y, \phi)\hat{R}(\boldsymbol{e}_x, \theta)\hat{R}(\boldsymbol{e}_z, \psi), \\ & \hat{R}(\boldsymbol{e}_z, \psi) = \cos\frac{\psi}{2} + \hat{k}\sin\frac{\psi}{2}, \\ & \hat{R}(\boldsymbol{e}_x, \theta) = \cos\frac{\theta}{2} + \hat{\imath}\sin\frac{\theta}{2}, \\ & \hat{R}(\boldsymbol{e}_y, \phi) = \cos\frac{\phi}{2} + \hat{\jmath}\sin\frac{\phi}{2}. \end{aligned}$$

A quaternion is an extension of complex numbers with the units $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, and 1, which follow the multiplication rules

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -1, \qquad \hat{\imath}\,\hat{\jmath} = -\hat{\jmath}\,\hat{\imath} = \hat{k}, \quad \hat{\jmath}\,\hat{k} = -\hat{k}\,\hat{\jmath} = \hat{\imath}, \quad \hat{k}\,\hat{\imath} = -\hat{\imath}\,\hat{k} = \hat{\jmath}.$$

5 Steady States and Stability Analysis

We look into steady solutions with the cocnstant angular momentum, and analyze their stability. To simplify the equations, we adopt a dimensionless unit system with $M = a = g = 1$, where

$$I_X = I_Y = \frac{1}{4}, \quad I_Z = \frac{1}{2},$$

and the equations of motion become

$$\begin{cases} \dot\omega_X &= -\frac{6}{5}\omega_Y\omega_Z - \frac{1}{5}\omega_Y^2\tan\theta + \frac{4}{5}\sin\theta \\ \dot\omega_Y &= \omega_X\omega_Y\tan\theta + 2\omega_Z\omega_X \\ \dot\omega_Z &= \frac{2}{3}\omega_X\omega_Y \end{cases} \qquad \begin{cases} \dot\theta &= \omega_X \\ \dot\phi &= \omega_Y\dfrac{1}{\cos\theta} \\ \dot\psi &= \omega_Z + \omega_Y\tan\theta \end{cases},$$

The steady solutions satisfy

$$\begin{cases} 0 &= -\frac{6}{5}\omega_Y\omega_Z - \frac{1}{5}\omega_Y^2\tan\theta + \frac{4}{5}\sin\theta \\ 0 &= \omega_X\omega_Y\tan\theta + 2\omega_Z\omega_X \\ 0 &= \frac{2}{3}\omega_X\omega_Y \end{cases},$$

from which, we obtain two conditions:

1. $$\omega_X = \omega_Y = \theta = 0, \quad \omega_Z = \text{ const.}$$

2. $$\omega_X = 0, \quad \left(\omega_Z,\ \theta\right) = \text{ const.}, \quad \omega_Y = f^{\pm}(\omega_Z,\theta) \equiv \frac{1}{\tan\theta}\left(-3\omega_Z \pm \sqrt{9\omega_Z^2 + 4\frac{\sin^2\theta}{\cos\theta}}\right)$$

   The function $f^{\pm}(\omega_Z,\theta)$ has

   $$f^{\pm}(\omega_Z,\theta) = -f^{\mp}(-\omega_Z,\theta) = f^{\mp}(-\omega_Z,-\theta)$$

   symmetry, so we may consider only $\omega_Z, \theta > 0$.

Steady State 1: Straight Rolling Motion

$$\omega_Z = \omega_{Z,0} \ = \text{const.}, \quad \omega_X = \omega_Y = \theta = 0$$

Steady State 2: Spinning with Stationary Center of Mass, and $\boldsymbol{F} = 0$, $\boldsymbol{R} = -M\boldsymbol{g}$

$$\omega_X = \omega_Z = 0, \quad \omega_Y = 2\sqrt{\cos\theta_0}, \quad \theta = \theta_0, \qquad \dot\phi = \frac{2}{\sqrt{\cos\theta_0}}$$

Radius of the contact point trajectory: R_C

$$R_C = \frac{\dot\psi}{\dot\phi} = \sin\theta_0$$

Steady State 3: Circular Motion

$$\omega_X = 0,\ \omega_Z = \omega_{Z,0},\ \theta = \theta_0, \quad \omega_Y = \omega_{Y,0} \equiv \frac{3\,\omega_{Z,0}}{\tan\theta_0}\left(-1 \pm \sqrt{1 + \frac{4}{9}\frac{\tan\theta_0}{\omega_{Z,0}^2}\sin\theta_0}\right)$$

Radius of COM trajectory: R_G,

$$R_G = \left|\frac{\omega_Z}{\dot\phi}\right| = \left|\frac{\omega_{Z,0}}{\omega_{Y,0}}\cos\theta_0\right| = \frac{3}{4}\frac{\omega_{Z,0}^2}{\tan\theta_0}\left(\sqrt{1 + \frac{4}{9}\frac{\tan\theta_0}{\omega_{Z,0}^2}\sin\theta_0} \pm 1\right)$$

Radius of the contact point trajectory: R_C

$$R_C = \left|\frac{\dot\psi}{\dot\phi}\right| = \left|\frac{\omega_{Z,0}}{\omega_{Y,0}}\cos\theta_0 + \sin\theta_0\right|$$
$$= \left|\frac{3}{4}\frac{\omega_{Z,0}^2}{\tan\theta_0}\left(\sqrt{1 + \frac{4}{9}\frac{\tan\theta_0}{\omega_{Z,0}^2}\sin\theta_0} \pm 1\right) \pm \sin\theta_0\right|$$

Steady State 4: Large Circular Motion

$$\omega_X = 0,\ \omega_Z = \omega_{Z,0},\ \theta = \theta_0 \ll 1$$

,

$$\omega_Y = \omega_{Y,0} \equiv \frac{3\,\omega_{Z,0}}{\tan\theta_0}\left(\sqrt{1 + \frac{4}{9}\frac{\tan\theta_0}{\omega_{Z,0}^2}\sin\theta_0} - 1\right) \approx \frac{2}{3}\frac{\sin\theta_0}{\omega_{Z,0}}$$

$$R_G = \frac{3}{4}\frac{\omega_{Z,0}^2}{\tan\theta_0}\left(\sqrt{1 + \frac{4}{9}\frac{\tan\theta_0}{\omega_{Z,0}^2}\sin\theta_0} + 1\right) \approx \frac{3}{2}\frac{\omega_{Z,0}^2}{\sin\theta_0} \approx \frac{\omega_{Z,0}}{\omega_{Y,0}}$$

5.1 Stability of Steady State 1:

$\omega_Z = \omega_{Z,0}$ = const., $\quad \omega_X = \omega_Y = \theta = 0$

Let small deviations from this state be

$$\omega_X = \delta\omega_X, \quad \omega_Y = \delta\omega_Y, \quad \omega_Z = \omega_{Z,0} + \delta\omega_Z, \quad \theta = \delta\theta$$

Then,

$$\begin{cases} \delta\dot\omega_X &= -\frac{6}{5}\omega_{Z,0}\delta\omega_Y + \frac{4}{5}\delta\theta \\ \delta\dot\omega_Y &= 2\omega_{Z,0}\delta\omega_X \\ \delta\dot\omega_Z &= 0 \end{cases} \qquad \begin{cases} \delta\dot\theta &= \delta\omega_X \\ \delta\dot\phi &= \delta\omega_Y \\ \dot\psi &= \omega_{Z,0} \end{cases}$$

Assuming $(\delta\omega_X, \delta\omega_Y, \delta\theta) = (A_X, A_Y, A_\theta)e^{\lambda t}$, we have

$$\lambda\boldsymbol{A} = \hat\Lambda\boldsymbol{A}\,; \qquad \hat\Lambda \equiv \begin{pmatrix} 0, & -\frac{6}{5}\omega_{Z,0}, & \frac{4}{5} \\ 2\omega_{Z,0}, & 0, & 0 \\ 1, & 0, & 0 \end{pmatrix}$$

Characteristic equation:

$$\lambda\left(\lambda^2 + \frac{12}{5}\omega_{Z,0}^2 - \frac{4}{5}\right) = 0$$

Eigenvalues:

$$\lambda = 0, \quad \pm\sqrt{\frac{4}{5}\left(1 - 3\omega_{Z,0}^2\right)},$$

from which the critical angular velocity is $\omega_c \equiv 1/\sqrt{3} \approx 0.577$.

Stability condition: Unstable if $\omega_{Z,0} < \omega_c$, and neutrally stable if $\omega_{Z,0} \geqq \omega_c$.

Neutral stability means that orbits of arbitrary radius exist, but with no restoring force, so even small perturbations result in transition to another circular trajectory.

5.2 Stability of Steady State 2:

Assume constants $\omega_X = 0$, and $(\omega_Y, \omega_Z, \theta) \neq 0$

$$I_X\omega_Y^2 \tan\theta + \left(I_Z + Ma^2\right)\omega_Y\omega_Z - Ma^2\frac{g}{a}\sin\theta = 0$$

satisfies the condition. Let the solution be:

$$\omega_X = \delta\omega_X, \quad \omega_Y = \omega_{Y,0} + \delta\omega_X, \quad \omega_Z = \omega_{Z,0} + \delta\omega_X, \quad \theta = \theta_0 + \delta\theta$$

Assuming:

6 Case of a Ring

Moments of inertia of a ring with outer radius a and inner radius b:

$$I_X = I_Y = \frac{1}{4}M(a^2 + b^2), \qquad I_Z = \frac{1}{2}M(a^2 + b^2)$$