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Dynamics of the Rattleback

The bottom surface of a rattleback can be approximated by a quadric surface, while its principal curvature directions deviate from the principal axes of inertia. To describe the dynamics of the rattleback, we introduce two coordinate systems: the coordinate system XYZ fixed to the rattleback with the axes aligned with the principal axes of inertia and its origin at the center of mass G, and the inertia coordinate system xyz with the xy plane on the floor and the z axis pointing upward vertical direction.

In the XYZ coordinate, the moment of inertia tensor $\hat{I}$ is diagonal and is assume to be given by

$$\hat{I} := \begin{pmatrix} A, & 0, & 0 \\ 0, & B, & 0 \\ 0, & 0, & C \end{pmatrix}.$$

On the other hand, the bottom surface of the rattleback is expressed by the shape function $f(\boldsymbol{R})$ as

$$f(\boldsymbol{R}) = 0.$$

The function $f(\boldsymbol{R})$ is given by a non-diagonal matrix $\hat{\Theta}_\xi$ as

$$f(\boldsymbol{R}) := -1 + \frac{1}{a^2}\big(X, Y, Z\big)\hat{\Theta}_\xi \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -1 + \frac{1}{a^2}\boldsymbol{R}^t\,\hat{\Theta}_\xi \boldsymbol{R}. \tag{1}$$

Let ξ be the misalignment angle between the principal axes of inertia and the principal axis of the ellipsoid. Then the matrix $\hat{\Theta}_\xi$ is given by

$$\hat{\Theta}_\xi := \hat{R}(\xi)\hat{\Theta}\hat{R}(\xi)^{-1}, \qquad \hat{\Theta} := \begin{pmatrix} \theta, & 0, & 0 \\ 0, & \phi, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad \hat{R}(\xi) := \begin{pmatrix} \cos\xi, & -\sin\xi, & 0 \\ \sin\xi, & \cos\xi, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$

The center of mass (COM) of the ellipsoid is assume to be located at the center of the ellipsoid, then the height of G (COM) is a when the rattleback is placed horizontally, and the principal curvatures are θ/a and ϕ/a. For a typical rattleback, these parameters satisfy the conditions

$$\xi \ll 1, \qquad A > B, \qquad 1 > \theta \gg \phi > 0.$$

Equations of Motion: Let $\boldsymbol{v}_G$ be the velocity of COM and $\boldsymbol{\omega}$ be the angular velocity, then the translation and the rotation equations of motion are give by

$$M\frac{d\boldsymbol{v}_G}{dt} = \boldsymbol{F} - Mg\hat{\boldsymbol{e}}_z \tag{2}$$

$$\frac{d\big(\hat{I}\boldsymbol{\omega}\big)}{dt} = \boldsymbol{R}_C \times \boldsymbol{F}; \quad \boldsymbol{R}_C := \boldsymbol{r}_C - \boldsymbol{r}_G. \tag{3}$$

Here, $\boldsymbol{F}$ is the reaction force from the floor, and $\boldsymbol{R}_C$ is the vector pointing from the center of mass G to the contact point with the floor C.

The position vector of the contact point $\boldsymbol{R}_C$ is determined by two conditions: the contact point C should be on the bottom surface, and the normal vector at the contact point C should be in the vertical direction, namely

$$f(\boldsymbol{R}_C) = 0, \qquad \boldsymbol{\nabla} f(\boldsymbol{R}_C) \parallel \hat{\boldsymbol{e}}_z. \tag{4}$$

The reaction force $\boldsymbol{F}$ determined by the no-slip condition,

$$\boldsymbol{v}_G = \boldsymbol{R}_C \times \boldsymbol{\omega}. \tag{5}$$

Solution of the Equations of Motion: Eliminating the reaction force $\boldsymbol{F}$ from the translation and the rotation equations of motion, we obtain

$$\frac{d(\hat{I}\boldsymbol{\omega})}{dt} = \boldsymbol{R}_C \times M\left(\frac{d}{dt}(\boldsymbol{R}_C \times \boldsymbol{\omega}) + g\hat{\boldsymbol{e}}_z\right).$$

Now, in the XYZ coordinate, the time derivative can be expressed as

$$\begin{aligned}
\frac{d(\hat{I}\boldsymbol{\omega})}{dt} &= \hat{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}), \\
\frac{d}{dt}(\boldsymbol{R}_C \times \boldsymbol{\omega}) &= \frac{\partial}{\partial t}(\boldsymbol{R}_C \times \boldsymbol{\omega}) + \boldsymbol{\omega} \times (\boldsymbol{R}_C \times \boldsymbol{\omega}) \\
&= \dot{\boldsymbol{R}}_C \times \boldsymbol{\omega} + \boldsymbol{R}_C \times \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{R}_C \times \boldsymbol{\omega}) \\
&= \dot{\boldsymbol{R}}_C \times \boldsymbol{\omega} + \boldsymbol{R}_C \times \dot{\boldsymbol{\omega}} + \omega^2\boldsymbol{R}_C - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{\omega},
\end{aligned}$$

using the apparent time derivative in the XYZ coordinate

$$\frac{\partial \boldsymbol{a}}{\partial t} := \dot{\boldsymbol{a}} := \dot{a}_X\hat{\boldsymbol{e}}_X + \dot{a}_Y\hat{\boldsymbol{e}}_Y + \dot{a}_Z\hat{\boldsymbol{e}}_Z$$

Then, the equation of motion is re-written as

$$\hat{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}) = M\boldsymbol{R}_C \times \left(\dot{\boldsymbol{R}}_C \times \boldsymbol{\omega} + \boldsymbol{R}_C \times \dot{\boldsymbol{\omega}} + \omega^2\boldsymbol{R}_C - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{\omega} + g\hat{\boldsymbol{e}}_z\right)$$

$$\begin{aligned}
\hat{I}\dot{\boldsymbol{\omega}} - M\boldsymbol{R}_C \times (\boldsymbol{R}_C \times \dot{\boldsymbol{\omega}}) &= -\boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}) + M\boldsymbol{R}_C \times \left(\dot{\boldsymbol{R}}_C \times \boldsymbol{\omega} + \omega^2\boldsymbol{R}_C - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{\omega} + g\hat{\boldsymbol{e}}_z\right) \\
\hat{I}\dot{\boldsymbol{\omega}} - M\left((\boldsymbol{R}_C \cdot \dot{\boldsymbol{\omega}})\boldsymbol{R}_C - R_C^2\dot{\boldsymbol{\omega}}\right) &= -\boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}) + M\left(\boldsymbol{R}_C \times (\dot{\boldsymbol{R}}_C \times \boldsymbol{\omega}) - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{R}_C \times \boldsymbol{\omega} + g\boldsymbol{R}_C \times \hat{\boldsymbol{e}}_z\right) \\
&= -\boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}) + M\left((\boldsymbol{R}_C \cdot \boldsymbol{\omega})\dot{\boldsymbol{R}}_C - (\boldsymbol{R}_C \cdot \dot{\boldsymbol{R}}_C)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{R}_C \times \boldsymbol{\omega} + g\boldsymbol{R}_C \times \hat{\boldsymbol{e}}_z\right) \\
\hat{I}_C\dot{\boldsymbol{\omega}} &= -\boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}) + M\left((\boldsymbol{R}_C \cdot \boldsymbol{\omega})\dot{\boldsymbol{R}}_C - (\boldsymbol{R}_C \cdot \dot{\boldsymbol{R}}_C)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{R}_C \times \boldsymbol{\omega} + g\boldsymbol{R}_C \times \hat{\boldsymbol{e}}_z\right)
\end{aligned}$$

$$\dot{\boldsymbol{\omega}} = \hat{I}_C^{-1}\left(-\boldsymbol{\omega} \times (\hat{I}\boldsymbol{\omega}) + M\left((\boldsymbol{R}_C \cdot \boldsymbol{\omega})\dot{\boldsymbol{R}}_C - (\boldsymbol{R}_C \cdot \dot{\boldsymbol{R}}_C)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{R}_C)\boldsymbol{R}_C \times \boldsymbol{\omega} + g\boldsymbol{R}_C \times \hat{\boldsymbol{e}}_z\right)\right)$$

Here, $\hat{I}_C$ is the inertia tensor centered around the contact point C,

$$\hat{I}_C := \begin{pmatrix} A + M(R_C^2 - X_C^2), & -MX_CY_C, & -MX_CZ_C \\ -MY_CX_C, & B + M(R_C^2 - Y_C^2), & -MY_CZ_C \\ -MZ_CX_C, & -MZ_CY_C, & C + M(R_C^2 - Z_C^2) \end{pmatrix}.$$

$$
\begin{cases}
\dfrac{d\boldsymbol{r}_G}{dt} &= \boldsymbol{R}_C \times \boldsymbol{\omega} \\
\dot{\boldsymbol{\omega}} &= \hat{I}_C^{-1}\Bigg(-\boldsymbol{\omega} \times \big(\hat{I}\boldsymbol{\omega}\big) \ + M\Big((\boldsymbol{R}_C \cdot \boldsymbol{\omega})\dot{\boldsymbol{R}}_C - (\boldsymbol{R}_C \cdot \dot{\boldsymbol{R}}_C)\boldsymbol{\omega} \\
& \qquad\qquad\qquad\qquad -\big(\boldsymbol{\omega} \cdot \boldsymbol{R}_C\big)\boldsymbol{R}_C \times \boldsymbol{\omega} + g\boldsymbol{R}_C \times \hat{\boldsymbol{e}}_z \Big) \Bigg) \\
\dfrac{dR_q}{dt} &= \dfrac{1}{2} R_q \omega_q \,; \qquad R_q: \ \text{the transformation quaternion from the } XYZ \text{ to the } xyz \text{ frame}
\end{cases}
$$

The contact point position $\boldsymbol{R}_C$ and its time derivative $\dot{\boldsymbol{R}}_C$

Suppose

$$f(\boldsymbol{R}) := -1 + \frac{1}{a^2}\big(X, Y, Z\big)\hat{\Theta}_\xi \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -1 + \frac{1}{a^2}\,\boldsymbol{R}^t\,\hat{\Theta}_\xi \boldsymbol{R},$$

$$\hat{\Theta}_\xi := \hat{R}(\xi)\hat{\Theta}\hat{R}(\xi)^{-1}, \qquad \hat{\Theta} := \begin{pmatrix} \theta, & 0, & 0 \\ 0, & \phi, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad \hat{R}(\xi) := \begin{pmatrix} \cos\xi, & -\sin\xi, & 0 \\ \sin\xi, & \cos\xi, & 0 \\ 0, & 0, & 1 \end{pmatrix},$$

then, we have

$$\boldsymbol{\nabla} f = \frac{2}{a^2}\,\hat{\Theta}_\xi \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{2}{a^2}\,\hat{\Theta}_\xi \boldsymbol{R},$$

which leads to

$$\boldsymbol{\nabla} f \parallel \hat{\boldsymbol{e}}_z \quad \Rightarrow \; \boldsymbol{e}_z = \frac{e_{zZ}}{Z_C}\,\hat{\Theta}_\xi \boldsymbol{R}_C \quad \Rightarrow \; \boldsymbol{R}_C = \frac{Z_C}{e_{zZ}}\,\hat{\Theta}_\xi^{-1} \boldsymbol{e}_z.$$

From $f(\boldsymbol{R}_C) = 0$, we also have

$$a^2 = \boldsymbol{R}_C^t\,\hat{\Theta}_\xi \boldsymbol{R}_C = \left(\frac{Z_C}{e_{zZ}}\right)^2 \boldsymbol{e}_z^t \hat{\Theta}_\xi^{-1} \boldsymbol{e}_z \quad \Rightarrow \quad Z_C = -a\, e_{zZ} \left(\boldsymbol{e}_z^t \hat{\Theta}_\xi^{-1} \boldsymbol{e}_z\right)^{-1/2}.$$

Time Derivative:

$$\dot{\boldsymbol{R}}_C = \left(\frac{\dot{Z}_C}{e_{zZ}} - \frac{Z_C \dot{e}_{zZ}}{e_{zZ}^2}\right)\hat{\Theta}_\xi^{-1}\hat{\boldsymbol{e}}_z + \frac{Z_C}{e_{zZ}}\,\hat{\Theta}_\xi^{-1}\dot{\boldsymbol{e}}_z$$

$$\dot{\boldsymbol{e}}_z = -\boldsymbol{\omega} \times \boldsymbol{e}_z$$

$$a^2 = \boldsymbol{R}_C^t\,\hat{\Theta}_\xi\,\boldsymbol{R}_C \quad \Rightarrow \; 0 = \dot{\boldsymbol{R}}_C^t\,\hat{\Theta}_\xi\,\boldsymbol{R}_C = \dot{\boldsymbol{R}}_C^t\,\frac{Z_C}{e_{zZ}}\boldsymbol{e}_z \quad \Rightarrow \; \boldsymbol{e}_z \cdot \dot{\boldsymbol{R}}_C = 0$$

Substituting the above expressions into this, we have

$$\left(\frac{\dot{Z}_C}{Z_C} - \frac{\dot{e}_{zZ}}{e_{zZ}}\right)\boldsymbol{e}_z^t\,\boldsymbol{R}_C = -\frac{Z_C}{e_{zZ}}\boldsymbol{e}_z^t\,\hat{\Theta}_\xi^{-1}\dot{\boldsymbol{e}}_z = -\boldsymbol{R}_C^t\dot{\boldsymbol{e}}_z$$

$$\therefore\; \dot{Z}_C = Z_C\left(\frac{\dot{e}_{zZ}}{e_{zZ}} - \frac{\dot{\boldsymbol{e}}_z^t\boldsymbol{R}_C}{\boldsymbol{e}_z^t\,\boldsymbol{R}_C}\right) = Z_C\left(\frac{\dot{e}_{zZ}}{e_{zZ}} - \frac{Z_C}{e_{zZ}a^2}\dot{\boldsymbol{e}}_z^t\boldsymbol{R}_C\right)$$

This matches the previous expression.

Parameters:

- Geometrical parameters:
$$a[\mathrm{L}] = 1, \quad \theta,\ \phi, \quad \xi$$
- Physical properties:
$$M[\mathrm{M}] = 1, \quad A,\ B,\ C,\ [\mathrm{ML}^2]$$
- Physical constants:
$$g[\mathrm{LT}^{-2}] = 1$$
- Auxiliary parameters:
$$L_x,\ r_x,\ L_y,\ r_y\ [\mathrm{L}], \quad \rho\ [\mathrm{ML}^{-3}]$$
$$\hat{I}, \quad \hat{\Theta},\ \hat{\Theta}_\xi, \quad R_\xi$$

Variables:

$$\boldsymbol{r}_G, \quad \boldsymbol{\omega}, \quad R_q$$
$$\boldsymbol{R}_C, \quad \boldsymbol{e}_z, \quad \hat{I}_C$$

mass and moment of inertia tensor

$$\begin{aligned}
&M = M_x + M_y = 1; \qquad M_x = \pi r_x^2 L_x \rho, \quad M_y = \pi r_y^2 L_y \rho \\
&A = \frac{1}{12} M_y L_y^2; \qquad M_y = \frac{r_y^2 L_y}{r_x^2 L_x + r_y^2 L_y}\, M \\
&B = \frac{1}{12} M_x L_x^2; \qquad M_x = \frac{r_x^2 L_x}{r_x^2 L_x + r_y^2 L_y}\, M \\
&C = A + B
\end{aligned}$$

Relationship between coordinate systems: $A_q^S = R_q A_q^B R_q^{-1}$

$$e_{zX}\hat{\imath} + e_{zY}\hat{\jmath} + e_{zZ}\hat{k} = R_q^{-1}\hat{k}\, R_q$$

$$\begin{aligned}
e_{Xx}\hat{\imath} + e_{Xy}\hat{\jmath} + e_{Xz}\hat{k} &= R_q\, \hat{\imath} R_q^{-1} \\
e_{Yx}\hat{\imath} + e_{Yy}\hat{\jmath} + e_{Yz}\hat{k} &= R_q\, \hat{\jmath} R_q^{-1} \\
e_{Zx}\hat{\imath} + e_{Zy}\hat{\jmath} + e_{Zz}\hat{k} &= R_q\, \hat{k} R_q^{-1}
\end{aligned}$$

Energy

$$E = K + U; \qquad K = \frac{1}{2} M v_G^2 + \frac{1}{2} \boldsymbol{\omega}^t \hat{I}\, \boldsymbol{\omega}, \quad U = -Mg\boldsymbol{R}_C \cdot \boldsymbol{e}_z$$

Small oscillations about the X axis (pitching):

Supposing that, in the case $\xi = 0$, the pitching oscillation satisfies

$$\boldsymbol{\omega} = (\omega, 0, 0), \quad \boldsymbol{R}_C \approx (0, Y_C, -a), \quad \boldsymbol{e}_z \approx (0, u, 1),$$

we will determine the period of small oscillation aroud the X axis within the first order approximation in ω, Y_C, and u.

The component of $\boldsymbol{e}_z$ in the XYZ frame follows the equation

$$\frac{d\boldsymbol{e}_z}{dt} = \dot{\boldsymbol{e}}_z + \boldsymbol{\omega} \times \boldsymbol{e}_z = 0 \quad \Rightarrow \quad \dot{u} \approx \omega, \tag{6}$$

while we can approximate as

$$Y_C \approx -a\phi^{-1}u, \qquad \dot{Y}_C \approx a\phi^{-1}\omega$$

because $Z_C \approx -a$, $e_{zZ} \approx 1$, $\dot{Z}_C \approx 0$, $\dot{e}_{zZ} \approx 0$.

These approximations lead to the equation for ω

$$\begin{aligned}\dot{\omega} &\approx \frac{M}{A + Ma^2}\, g\,(0, Y_C, -a) \times (0, u, 1)\big|_X = \frac{Ma^2}{A + Ma^2}\,\frac{g}{a^2}\,\big(Y_C + au\big) \\ &= -\frac{Ma^2}{A + Ma^2}\,\frac{g}{a}\,\big(\phi^{-1} - 1\big)u\end{aligned}$$

within the first order approximation of small quantities. Using Eq.(6), we have

$$\ddot{\omega} \approx -\frac{Ma^2}{A + Ma^2}\,\frac{g}{a}\,\big(\phi^{-1} - 1\big)\omega,$$

thus the pitching oscillation period is

$$T_{\text{pitch}} \approx 2\pi\sqrt{\left(\frac{A}{Ma^2} + 1\right)\frac{1}{\phi^{-1} - 1}\,\frac{a}{g}}.$$

Small oscillations about the Y axis (rolling):

Similarly, the rolling oscillation period is

$$T_{\text{roll}} \approx 2\pi\sqrt{\left(\frac{B}{Ma^2} + 1\right)\frac{1}{\theta^{-1} - 1}\,\frac{a}{g}}.$$