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Huygens' Synchronization of Coupled Two Pendula

We will try to invent simplest model for synchronization of two clocks, or Huygens synchronization.

1) Single pendulum: The clock may be modeled by a pendulum with damping and excitation. First, we formulate the pendulum without damping and excitation:

$$x = \ell \sin\theta, \quad y = -\ell\cos\theta, \qquad \dot{x} = \ell\cos\theta\,\dot{\theta}, \quad \dot{y} = \ell\sin\theta\,\dot{\theta}.$$

Then, the Lagrangian

$$L = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

the equations of motion

$$\begin{cases} p_\theta &= \dfrac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \\ \dfrac{dp_\theta}{\partial t} &= -mg\ell\sin\theta. \end{cases}$$

With the damping and the exitation, the equations of motion become

$$\begin{cases} \dot{\theta} &= \dfrac{1}{m\ell^2}\,p_\theta, \\ \dot{p}_\theta &= -mg\ell\sin\theta - \dfrac{1}{\tau}\,p_\theta + N(\theta)\dfrac{p_\theta}{|p_\theta|} \end{cases}$$

with the exitation function $N(\theta)$, which may be something like

$$N(\theta) = N_0\frac{1}{1+(\theta/\theta_0)^4}.$$

2) Coupled pendula: Two pendula are coupled through the board with the mass m_0 and the elastic constant K.

the coordinates:

$$x_1 = X + \ell_1\sin\theta_1, \quad y_1 = -\ell_1\cos\theta_1, \quad x_2 = X + \ell_2\sin\theta_2, \quad y_2 = -\ell_2\cos\theta_2, \quad X$$

the velocities:

$$\dot{x}_1 = \dot{X} + \dot{\theta}_1\ell_1\cos\theta_1, \quad \dot{y}_1 = \dot{\theta}_1\ell_1\sin\theta_1, \quad \dot{x}_2 = \dot{X} + \dot{\theta}_2\ell_2\cos\theta_2, \quad \dot{y}_2 = \dot{\theta}_2\ell_2\sin\theta_2, \quad \dot{X}$$

Lagrangian:

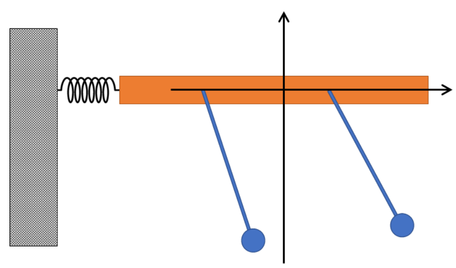
$$\begin{aligned} L &= T - U \\ T &= \frac{1}{2}m_1\Big(\dot{X}^2 + 2\ell_1\cos\theta_1\dot{X}\dot{\theta}_1 + \ell_1^2\dot{\theta}_1^2\Big) + \frac{1}{2}m_2\Big(\dot{X}^2 + 2\ell_2\cos\theta_2\dot{X}\dot{\theta}_2 + \ell_2^2\dot{\theta}_2^2\Big) + \frac{1}{2}m_0\dot{X}^2 \\ U &= -m_1 g\ell_1\cos\theta_1 - m_2 g\ell_2\cos\theta_2 + \frac{1}{2}KX^2 \end{aligned}$$

Momenta:

$$
\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{\theta}_1} = m_1\Big(\ell_1 \cos\theta_1 \dot{X} + \ell_1^2 \dot{\theta}_1\Big) \\
p_2 &= \frac{\partial L}{\partial \dot{\theta}_2} = m_2\Big(\ell_2 \cos\theta_2 \dot{X} + \ell_2^2 \dot{\theta}_2\Big) \\
p_X &= \frac{\partial L}{\partial \dot{X}} = \big(m_1 + m_2 + m_0\big)\dot{X} + m_1\ell_1 \cos\theta_1\, \dot{\theta}_1 + m_2\ell_2 \cos\theta_2\, \dot{\theta}_2
\end{aligned}
$$

Equations of Motion:

$$
\begin{aligned}
\dot{p}_1 &= \frac{\partial L}{\partial \theta_1} = -\big(m_1\ell_1\dot{X}\dot{\theta}_1 + m_1 g\ell_1\big)\sin\theta_1 - \frac{1}{\tau_1}\, p_1 + N_1(\theta_1)\, \frac{p_1}{|p_1|} \\
\dot{p}_2 &= \frac{\partial L}{\partial \theta_2} = -\big(m_2\ell_2\dot{X}\dot{\theta}_2 + m_2 g\ell_2\big)\sin\theta_2 - \frac{1}{\tau_2}\, p_2 + N_2(\theta_2)\, \frac{p_2}{|p_2|} \\
\dot{p}_X &= \frac{\partial L}{\partial X} = -KX - \frac{1}{\tau_X}\, p_X
\end{aligned}
$$



$$L=\frac{1}{2}\dot{\boldsymbol{q}}^t\hat{A}\dot{\boldsymbol{q}}-U,\quad \boldsymbol{p}=\hat{A}\dot{\boldsymbol{q}},\qquad H=\frac{1}{2}\boldsymbol{p}^t\hat{A}^{-1}\boldsymbol{p}+U(\boldsymbol{q})$$

$$\boldsymbol{q}=\begin{pmatrix}\theta_1\\ \theta_2\\ X\end{pmatrix},\quad \boldsymbol{p}=\begin{pmatrix}p_1\\ p_2\\ p_X\end{pmatrix},$$

$$\hat{A}=\begin{pmatrix} m_1\ell_1^2, & 0, & m_1\ell_1\cos\theta_1\\ 0, & m_2\ell_2^2, & m_2\ell_2\cos\theta_2\\ m_1\ell_1\cos\theta_1, & m_2\ell_2\cos\theta_2, & M\end{pmatrix}\equiv\begin{pmatrix} a, & 0, & d\\ 0, & b, & e\\ d, & e, & c\end{pmatrix}$$

$$M\equiv m_1+m_2+m_0$$

$$\hat{A}^{-1}=\frac{1}{\det}\begin{pmatrix} bc-e^2, & ed, & -bd\\ ed, & ac-d^2, & -ae\\ -bd, & -ae, & ab\end{pmatrix}\equiv\frac{1}{\det}\begin{pmatrix} a', & f', & d'\\ f', & b', & e'\\ d', & e', & c'\end{pmatrix}$$

$$\begin{aligned}\det &= abc-ae^2-bd^2\\ &= m_2\ell_2^2\, m_2\ell_2^2\, M-m_1\ell_1^2\big(m_2\ell_2\cos\theta_2\big)^2-m_2\ell_2^2\big(m_1\ell_1\cos\theta_1\big)^2\end{aligned}$$

Equations of Motion:

$$\dot{\boldsymbol{q}}=\frac{\partial H}{\partial \boldsymbol{p}}=\hat{A}^{-1}\boldsymbol{p},\qquad \dot{\boldsymbol{p}}=-\frac{\partial H}{\partial \boldsymbol{q}}$$

$$\begin{cases}\dot\theta_1 &= \dfrac{1}{\det}\big(a'p_1+f'p_2+d'p_X\big)\\ \dot\theta_2 &= \dfrac{1}{\det}\big(f'p_1+b'p_2+e'p_X\big)\\ \dot X &= \dfrac{1}{\det}\big(d'p_1+e'p_2+c'p_X\big)\end{cases}$$

$$\begin{cases}\dot p_1 &= -\big(m_1\ell_1\dot X\dot\theta_1+m_1g\ell_1\big)\sin\theta_1-\dfrac{1}{\tau_1}\,p_1+N_1(\theta_1)\,\dfrac{p_1}{|p_1|}\\ \dot p_2 &= -\big(m_2\ell_2\dot X\dot\theta_2+m_2g\ell_2\big)\sin\theta_2-\dfrac{1}{\tau_2}\,p_2+N_2(\theta_2)\,\dfrac{p_2}{|p_2|}\\ \dot p_X &= -KX-\dfrac{1}{\tau_X}\,p_X\end{cases}$$

3) Coupled three pendula: Three pendula are coupled through the board with the mass m_0 and the elastic constant K.

the coordinates and velocities:

$$\begin{cases} x_1 &= X + \ell_1 \sin\theta_1 \\ y_1 &= -\ell_1 \cos\theta_1 \end{cases} \qquad \begin{cases} \dot{x}_1 &= \dot{X} + \dot{\theta}_1 \ell_1 \cos\theta_1, \\ \dot{y}_1 &= \dot{\theta}_1 \ell_1 \sin\theta_1, \end{cases}$$

$$\begin{cases} x_2 &= X + \ell_2 \sin\theta_2 \\ y_2 &= -\ell_2 \cos\theta_2 \end{cases} \qquad \begin{cases} \dot{x}_2 &= \dot{X} + \dot{\theta}_2 \ell_2 \cos\theta_2, \\ \dot{y}_2 &= \dot{\theta}_2 \ell_2 \sin\theta_2, \end{cases}$$

$$\begin{cases} x_3 &= X + \ell_3 \sin\theta_3 \\ y_3 &= -\ell_3 \cos\theta_3 \end{cases} \qquad \begin{cases} \dot{x}_3 &= \dot{X} + \dot{\theta}_3 \ell_3 \cos\theta_3, \\ \dot{y}_3 &= \dot{\theta}_3 \ell_3 \sin\theta_3, \end{cases}$$

$$X \qquad\qquad \dot{X}$$

Lagrangian:

$$L = T - U$$

$$T = \frac{1}{2} m_1 \left(\dot{X}^2 + 2\ell_1 \cos\theta_1 \dot{X}\dot{\theta}_1 + \ell_1^2 \dot{\theta}_1^2 \right) \quad + \frac{1}{2} m_2 \left(\dot{X}^2 + 2\ell_2 \cos\theta_2 \dot{X}\dot{\theta}_2 + \ell_2^2 \dot{\theta}_2^2 \right)$$

$$+ \frac{1}{2} m_3 \left(\dot{X}^2 + 2\ell_3 \cos\theta_3 \dot{X}\dot{\theta}_3 + \ell_3^2 \dot{\theta}_3^2 \right) \quad + \frac{1}{2} m_0 \dot{X}^2$$

$$U = -m_1 g \ell_1 \cos\theta_1 - m_2 g \ell_2 \cos\theta_2 - m_3 g \ell_3 \cos\theta_3 + \frac{1}{2} K X^2$$

Momenta:

$$p_1 = \frac{\partial L}{\partial \dot{\theta}_1} = m_1 \left(\ell_1 \cos\theta_1 \dot{X} + \ell_1^2 \dot{\theta}_1 \right)$$

$$p_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 \left(\ell_2 \cos\theta_2 \dot{X} + \ell_2^2 \dot{\theta}_2 \right)$$

$$p_3 = \frac{\partial L}{\partial \dot{\theta}_3} = m_3 \left(\ell_3 \cos\theta_3 \dot{X} + \ell_3^2 \dot{\theta}_3 \right)$$

$$p_X = \frac{\partial L}{\partial \dot{X}} = \left(m_1 + m_2 + m_3 + m_0 \right) \dot{X} + m_1 \ell_1 \cos\theta_1 \, \dot{\theta}_1 + m_2 \ell_2 \cos\theta_2 \, \dot{\theta}_2 + m_3 \ell_3 \cos\theta_3 \, \dot{\theta}_3$$

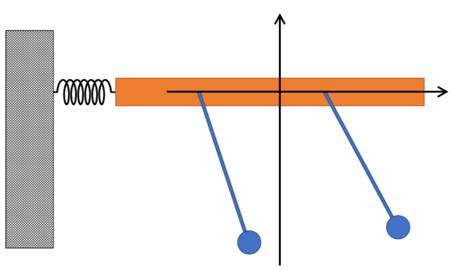
Equations of Motion:

$$\dot{p}_1 = \frac{\partial L}{\partial \theta_1} = -\left(m_1 \ell_1 \dot{X} \dot{\theta}_1 + m_1 g \ell_1 \right) \sin\theta_1 - \frac{1}{\tau_1} p_1 + N_1(\theta_1) \frac{p_1}{|p_1|}$$

$$\dot{p}_2 = \frac{\partial L}{\partial \theta_2} = -\left(m_2 \ell_2 \dot{X} \dot{\theta}_2 + m_2 g \ell_2 \right) \sin\theta_2 - \frac{1}{\tau_2} p_2 + N_2(\theta_2) \frac{p_2}{|p_2|}$$

$$\dot{p}_3 = \frac{\partial L}{\partial \theta_3} = -\left(m_3 \ell_3 \dot{X} \dot{\theta}_3 + m_3 g \ell_3 \right) \sin\theta_3 - \frac{1}{\tau_3} p_3 + N_3(\theta_3) \frac{p_3}{|p_3|}$$

$$\dot{p}_X = \frac{\partial L}{\partial X} = -KX - \frac{1}{\tau_X} p_X$$



$$L = \frac{1}{2}\dot{\boldsymbol{q}}^t \hat{A} \dot{\boldsymbol{q}} - U, \quad \boldsymbol{p} = \hat{A}\dot{\boldsymbol{q}}, \qquad H = \frac{1}{2}\boldsymbol{p}^t \hat{A}^{-1}\boldsymbol{p} + U(\boldsymbol{q})$$

$$\boldsymbol{q} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ X \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_X \end{pmatrix},$$

$$\hat{A} = \begin{pmatrix} m_1\ell_1^2, & 0, & 0, & m_1\ell_1\cos\theta_1 \\ 0, & m_2\ell_2^2, & 0, & m_2\ell_2\cos\theta_2 \\ 0, & 0, & m_3\ell_3^2, & m_3\ell_3\cos\theta_3 \\ m_1\ell_1\cos\theta_1, & m_2\ell_2\cos\theta_2, & m_3\ell_3\cos\theta_3, & M \end{pmatrix}$$

$$M \equiv m_1 + m_2 + m_3 + m_0$$

Equations of Motion:

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} = \hat{A}^{-1}\boldsymbol{p}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}$$

$$\begin{cases} \dot{p}_1 &= -\left(m_1\ell_1\dot{X}\dot{\theta}_1 + m_1 g\ell_1\right)\sin\theta_1 - \frac{1}{\tau_1}\,p_1 + N_1(\theta_1)\,\frac{p_1}{|p_1|} \\ \dot{p}_2 &= -\left(m_2\ell_2\dot{X}\dot{\theta}_2 + m_2 g\ell_2\right)\sin\theta_2 - \frac{1}{\tau_2}\,p_2 + N_2(\theta_2)\,\frac{p_2}{|p_2|} \\ \dot{p}_3 &= -\left(m_3\ell_3\dot{X}\dot{\theta}_3 + m_3 g\ell_3\right)\sin\theta_3 - \frac{1}{\tau_3}\,p_3 + N_3(\theta_3)\,\frac{p_3}{|p_3|} \\ \dot{p}_X &= -KX - \frac{1}{\tau_X}\,p_X \end{cases}$$

Double Pendulum

Centers of mass:

$$x_1 = \frac{1}{2}\ell_1 \sin\theta_1, \quad y_1 = -\frac{1}{2}\ell_1 \cos\theta_1, \qquad x_2 = \ell_1 \sin\theta_1 + \frac{1}{2}\ell_2 \sin\theta_2, \quad y_2 = -\ell_1 \cos\theta_1 - \frac{1}{2}\ell_2 \cos\theta_2$$

$$\dot{x}_1 = \frac{1}{2}\ell_1 \cos\theta_1 \dot{\theta}_1, \quad \dot{y}_1 = \frac{1}{2}\ell_1 \sin\theta_1 \dot{\theta}_1, \qquad \dot{x}_2 = \ell_1 \cos\theta_1 \dot{\theta}_1 + \frac{1}{2}\ell_2 \cos\theta_2 \dot{\theta}_2, \quad \dot{y}_2 = \ell_1 \sin\theta_1 \dot{\theta}_1 + \frac{1}{2}\ell_2 \sin\theta_2 \dot{\theta}_2.$$

Kinetic Energy:

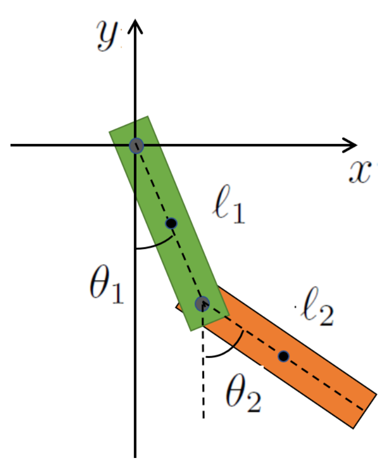
$$\begin{aligned}
T &= \frac{1}{2}m_1\left(\frac{1}{2}\ell_1\dot{\theta}_1\right)^2 + \frac{1}{2}L_1\dot{\theta}_1^2 \\
&\quad + \frac{1}{2}m_2\left(\left(\ell_1\cos\theta_1\dot{\theta}_1 + \frac{1}{2}\ell_2\cos\theta_2\dot{\theta}_2\right)^2 + \left(\ell_1\sin\theta_1\dot{\theta}_1 + \frac{1}{2}\ell_2\sin\theta_2\dot{\theta}_2\right)^2\right) + \frac{1}{2}L_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_1\left(\frac{1}{2}\ell_1\dot{\theta}_1\right)^2 + \frac{1}{2}L_1\dot{\theta}_1^2 + \frac{1}{2}m_2\left(\left(\ell_1\dot{\theta}_1\right)^2 + \ell_1\dot{\theta}_1\ell_2\dot{\theta}_2\cos\left(\theta_1-\theta_2\right) + \left(\frac{1}{2}\ell_2\dot{\theta}_2\right)^2\right) + \frac{1}{2}L_2\dot{\theta}_2^2 \\
&= \frac{1}{2}\left(m_1\left(\frac{1}{2}\ell_1\right)^2 + L_1 + m_2\ell_1^2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_1\ell_2\cos\left(\theta_1-\theta_2\right)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\left(m_2\left(\frac{1}{2}\ell_2\right)^2 + L_2\right)\dot{\theta}_2^2 \\
&= \frac{1}{2}\left(\dot{\theta}_1, \dot{\theta}_2\right)\begin{pmatrix} \left(m_1\left(\frac{1}{2}\ell_1\right)^2 + L_1 + m_2\ell_1^2\right), & \frac{1}{2}m_2\ell_1\ell_2\cos\left(\theta_1-\theta_2\right) \\ \frac{1}{2}m_2\ell_1\ell_2\cos\left(\theta_1-\theta_2\right), & \left(m_2\left(\frac{1}{2}\ell_2\right)^2 + L_2\right)\end{pmatrix}\begin{pmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{pmatrix} \equiv \frac{1}{2}\dot{\boldsymbol{\theta}}^t\hat{A}\dot{\boldsymbol{\theta}}
\end{aligned}$$

Potential Energy:

$$U = -m_1 g\frac{1}{2}\ell_1\cos\theta_1 - m_2 g\left(\ell_1\cos\theta_1 + \frac{1}{2}\ell_2\cos\theta_2\right)$$

Hamiltonian:

$$H = \frac{1}{2}\boldsymbol{p}^t\hat{A}^{-1}\boldsymbol{p} + U(\boldsymbol{\theta}); \qquad \dot{\boldsymbol{\theta}} = \hat{A}^{-1}\boldsymbol{p}, \quad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial\boldsymbol{\theta}}.$$



Equations of Motion:

$$
\begin{aligned}
\dot{\boldsymbol{\theta}} &= \hat{A}^{-1}\boldsymbol{p} \\
\dot{p}_1 &= \frac{\partial L}{\partial \theta_1} = -\frac{1}{2}m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - m_1 g\frac{1}{2}\ell_1\sin\theta_1 - m_2 g\ell_1\sin\theta_1 \\
\dot{p}_2 &= \frac{\partial L}{\partial \theta_2} = +\frac{1}{2}m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - m_2 g\frac{1}{2}\ell_2\sin\theta_2
\end{aligned}
$$

$$
\begin{aligned}
\hat{A} &= \begin{pmatrix} \left(m_1\left(\frac{1}{2}\ell_1\right)^2 + L_1 + m_2\ell_1^2\right), & \frac{1}{2}m_2\ell_1\ell_2\cos\left(\theta_1-\theta_2\right) \\ \frac{1}{2}m_2\ell_1\ell_2\cos\left(\theta_1-\theta_2\right), & \left(m_2\left(\frac{1}{2}\ell_2\right)^2 + L_2\right) \end{pmatrix} \equiv \begin{pmatrix} a, & c \\ c, & b \end{pmatrix} \\
\hat{A}^{-1} &= \frac{1}{\det}\begin{pmatrix} b, & -c \\ -c, & a \end{pmatrix}, \qquad \det \equiv ab - c^2
\end{aligned}
$$